Problem 5. Hawks and Doves

General assumptions

Let us consider a large population of players. Two of them are randomly chosen at each moment of time, and they start interacting with each other. Each of the two players has a right to choose one of the two strategies within this interaction. For our purposes the first strategy can be described as “passive”, whereas the second one – “aggressive”. A history of previous interactions is available to all players, which means that every participant of every interaction knows about the statistics of strategic choices. However, when players start their interactions, they do not know if they ever met before and what strategy their counter-agent chose. In other words, we consider anonymous game. Players choose their strategy, be it passive or aggressive, independently and simultaneously, on the basis of the payoffs they expect. Therefore, we can observe the process of learning in the game.

There are two types of games: symmetric and asymmetric. ***Symmetric game*** presupposes interaction between identical players that can be described as a player and its copy (opponent). In ***asymmetric game*** players are assigned with different roles, and their strategies vary according to these roles.

Problem statement

Setting off for a search of the twelve chairs, Ostap Bender and Kisa Vorobyaninov have a very limited amount of money at their disposal. They can only afford to buy one pepperoni sandwich each morning for breakfast. They constantly argue about how to share these sandwiches. Each of them has two strategies: passive and aggressive.

On one hand, Ostap Bender can show mercy on his elderly friend Kisa and let him eat the whole sandwich. On the other hand, Ostap can fight for the sandwich. Kisa, in his turn, has exact same strategies: either to give the whole sandwich to Ostap, or to fight for the right to eat it all by himself.

If both of them wake up in a good mood, ready to give a morning sandwich away, they divide it in two parts, and each of them receives utility equal to 3. If one wants to give his breakfast away and the other – to fight, the latter takes the whole sandwich and his utility equal to 6. If they both decide to fight, they ruin the sandwich, so both of them receives negative utility equal to (– 1) (as they do not eat and spend their power fighting).

Task 1

Let us consider **the symmetric game**.

1. Write down equations of evolutionary dynamics for Ostap Bender and for Kisa Vorobyaninov.
2. Find all of the equilibria that can be set in this interaction between Kisa and Ostap.
3. Test the stability of obtained equilibria.

Solution 1

1. Let us draw the game matrix.

If the game is symmetric, Ostap “competes” with Kisa, who is, in abstract terms, his copy. The probability of choosing a passive strategy is the same for both of them.

		Kisa Vorobyaninov	
		Passive Strategy (p)	Aggressive Strategy ($1-p$)
Ostap Bender	Passive Strategy (p)	**3; 3**	**0; 6**

	Aggressive Strategy $(1-p)$	**6; 0**	**-1; -1**

Let us write down the evolutionary dynamics equation for Ostap Bender:

$$\frac{dp}{dt} = p(U_{passive} - \bar{U}),$$

where $U_{passive}$ is Ostap's payoff which he gets if he chooses to a passive strategy, and $\bar{U}$ is the expected payoff of both strategies that are calculated as follows:

$$U_{passive} = 3p + 0 \cdot (1-p) = 3p$$
$$U_{agressive} = 6p + (-1) \cdot (1-p) = 7p - 1$$
$$\bar{U} = pU_{passive} + (1-p)U_{agressive} = p3p + (1-p)(7p-1)$$

Subsequently, the dynamics equation for Ostap Bender becomes as follows:

$$\frac{dp}{dt} = p(3p - 3p^2 - (7p-1)(1-p)) = p(1-p)(1-4p)$$

As this game is symmetric, the evolutionary dynamics equation for Kisa Vorobyaninov looks similar to the one for Ostap Bender.

2. In the equilibrium $\frac{dp}{dt} = 0$, and subsequently, the probability of a choice of a passive strategy does not change with time.
Let us draw the equilibrium probabilities:

$$\frac{dp}{dt} = p(1-p)(1-4p) = 0.$$

Subsequently, the equilibrium probabilities are $p = 0$, $p = 1$, $p = \frac{1}{4}$. Thus, there are three possible equilibria: : <0; 0>, <1; 1>**,** $<\frac{1}{4}$**;** $\frac{1}{4}>$. First comes the equilibrium probability for Ostap Bender, and second – for Kisa Vorobyaninov.

3. Let us test the found equilibria for stability.

Theorem. If $\frac{dp}{dt} = 0$ and $\frac{d\dot{p}}{dp} = \frac{d(\frac{dp}{dt})}{dp} < 0$, then p is an evolutionally stable strategy.
For this interaction we have:

$$\frac{d\dot{p}}{dp} = 12p^2 - 10p + 1.$$

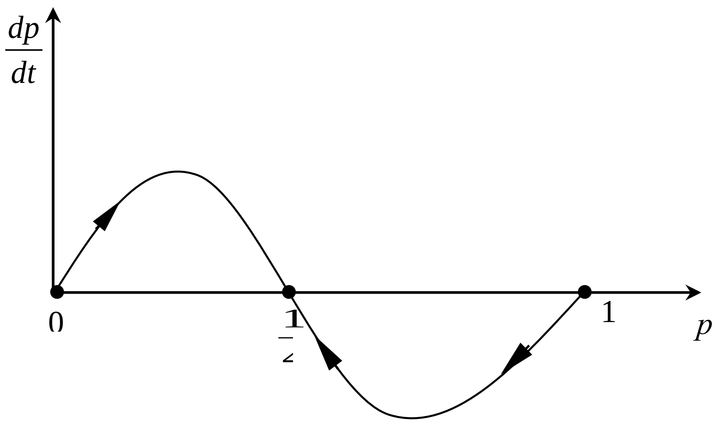
Let us test the found equilibria for stability:

$$\left.\frac{d\left(\frac{dp}{dt}\right)}{dp}\right|_{p=0} = 12\cdot(0)^2 - 10\cdot 0 + 1 = 1 > 0$$

$$\left.\frac{d\left(\frac{dp}{dt}\right)}{dp}\right|_{p=1} = 12\cdot(1)^2 - 10\cdot 1 + 1 = 3 > 0$$

$$\left.\frac{d\left(\frac{dp}{dt}\right)}{dp}\right|_{p=\frac{1}{4}} = 12\cdot(0{,}25)^2 - 10\cdot 0{,}25 + 1 = -\frac{3}{4} < 0$$

We observe that only the equilibrium $< \frac{1}{4}\ ;\ \frac{1}{4} >$ is stable, which means that with a probability $p = \frac{1}{4}$ Ostap Bender will choose to give the sandwich away, will choose a passive strategy, and with a $\frac{3}{4}$ probability he will choose to fight, choose an aggressive strategy. All this is equally true for Kisa Vorobyaninov. The phase diagram is drawn on picture 1.



Picture 1. Phase diagram of a game

The instability of the equilibria <0; 0>, <1; 1> can be explained in the following way. If one of the players accidentally diverges from the equilibrium, it will get a higher payoff than if he stays within the equilibrium, and it will not be beneficial to come back. Moreover, such behavior will affect the strategic choice of the opponent. Subsequently, the players will both deviate from the equilibria.

Task 2

Let us consider **the asymmetric game**.

1. Write down equations of evolutionary dynamics for Ostap Bender and for Kisa Vorobyaninov.
2. Find all of the equilibria that can be set in this interaction.
3. Test the stability of obtained equilibria.

Solution 2

1. The set of strategies and the payoffs remain the same as in the previous case. The equilibrium probabilities will we written as $< p;\ q >$ where p is a probability of Ostap Bender's choice of a passive strategy, and q is a probability of Kisa Vorobyaninov's choice of a passive strategy. Let us draw the game matrix for this case.

		Kisa Vorobyaninov	
		Passive (q)	Aggressive ($1 - q$)
Ostap Bender	Passive (p)	**3; 3**	**0; 6**
	Aggressive ($1 - p$)	**6; 0**	**-1; -1**

Let us write down the evolutionary dynamics equations for both players.

Ostap Bender

$$\frac{dp}{dt} = p(U_{passive} - \overline{U})$$
$$U_1 = 3q + 0 \cdot (1-q)$$
$$U_2 = 6q + (-1)(1-q) = 7q - 1$$
$$\overline{U} = pU_{passive} + (1-p)U_{agressive} =$$
$$= p3q + (1-p)(7q-1) =$$
$$= 7q - 4pq + p - 1$$
$$\frac{dp}{dt} = p(1-p)(1-4q)$$

Kisa Vorobyaninov

$$\frac{dq}{dt} = q(U_{passive} - \overline{U})$$
$$U_1 = 3p + 0 \cdot (1-p)$$
$$U_2 = 6p + (-1)(1-p) = 7p - 1$$
$$\overline{U} = qU_{passive} + (1-q)U_{agressive} =$$
$$= q3p + (1-q)(7p-1) =$$
$$= 7p - 4pq + q - 1$$
$$\frac{dq}{dt} = q(1-q)(1-4p)$$

2. Let us find the equilibria that can be set in this game. In order to do this, let us solve the system of equations.

$$\begin{cases} \dfrac{dp}{dt} = p(U_{passive} - \bar{U}) = p(1-p)(1-4q) = 0 \\ \dfrac{dq}{dt} = q(U_{passive} - \bar{U}) = q(1-q)(1-4p) = 0 \end{cases}$$

As a result, there are five possible equilibria: $<p;q>$: <0;1>; <1;0>; <1;1>; <0;0>; $<\frac{1}{4};\ \frac{1}{4}>$.

3. Let us test the found equilibria for stability. A stable equilibrium is an equilibrium that satisfies the following two conditions:

$$\frac{d\dot{p}}{dp} = \frac{d(\frac{dp}{dt})}{dp} < 0$$

and

$$\frac{d\dot{q}}{dq} = \frac{d(\frac{dq}{dt})}{dq} < 0 \ .$$

The described interaction is:

$$\begin{cases} \dfrac{d\dot{p}}{dp} = (1-4q)(1-2p) \\ \dfrac{d\dot{q}}{q} = (1-4p)(1-2q) \end{cases}$$

If we put the found equilibria to the system of equations considered above, we get:

1. $$\begin{cases} \left.\dfrac{d\dot{p}}{dp}\right|_{\substack{p=0\\q=0}} = 1 > 0 \\ \left.\dfrac{d\dot{q}}{dq}\right|_{\substack{p=0\\q=0}} = 1 > 0 \end{cases}$$

2. $$\begin{cases} \left.\dfrac{d\dot{p}}{dp}\right|_{\substack{p=1\\q=0}} = -1 < 0 \\ \left.\dfrac{d\dot{q}}{dq}\right|_{\substack{p=1\\q=0}} = -3 < 0 \end{cases}$$

3. $$\begin{cases} \left.\dfrac{d\dot{p}}{dp}\right|_{\substack{p=1\\q=1}} = 3 > 0 \\ \left.\dfrac{d\dot{q}}{dq}\right|_{\substack{p=1\\q=1}} = 3 > 0 \end{cases}$$

4. $$\begin{cases} \left.\dfrac{d\dot{p}}{dp}\right|_{\substack{p=0\\q=1}} = -3 < 0 \\ \left.\dfrac{d\dot{q}}{dq}\right|_{\substack{p=0\\q=1}} = -1 < 0 \end{cases}$$

5. $$\begin{cases} \left.\dfrac{d\dot{p}}{dp}\right|_{\substack{p=\frac{1}{4}\\q=\frac{1}{4}}} = 0 \\ \left.\dfrac{d\dot{q}}{dq}\right|_{\substack{p=\frac{1}{4}\\q=\frac{1}{4}}} = 0 \end{cases}$$

As a result, the equilibria <0;1> and <1;0> (2 and 4) are stable.

Let us analyze variation of a probability of a passive strategy in case a counter-agent deviates from the equilibrium.

Let us agree that $q > \frac{1}{4}$.

$$\frac{dp}{dt} = p(1-p)(1-4q)\ \Big|_{q>\frac{1}{4}} < 0$$

The higher Kisa’s probability to choose a passive strategy is, the lower the same probability is for Ostap, and the higher Ostap’s probability to choose an aggressive strategy is. In other words, if one of the players choses to be passive and to share, the other player is more likely to fight and get a higher utility.

The same is for $q < \frac{1}{4}$.

$$\frac{dp}{dt} = p(1-p)(1-4q)\ \Big|_{q<\frac{1}{4}} > 0$$

The lower Kisa’s probability to act passively and give a sandwich away is, the more likely Ostap is to choose to be passive, not to fight, as this way he will gain more benefit.

If we analyze the choice of Kisa with regard of changes in Ostap’s behavior, we will see that it matches with Ostap’s strategy described above. If a player’s opponent frequently chooses to fight, the player will choose a passive strategy, not to have a breakfast, to get a 0 benefit, but at least not to spend power on a fight (which means a negative benefit). If a player's opponent frequently chooses a passive strategy, the player will tend to act aggressively to increase his payoff. All this can be written down as follows.

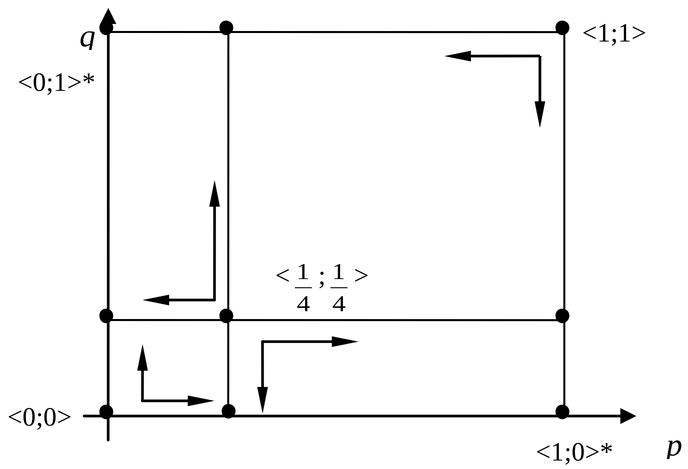
For Ostap Bender: If $q > \frac{1}{4} \Rightarrow \frac{dp}{dt} < 0 \Rightarrow p \downarrow$

$$\text{If } q < \frac{1}{4} \Rightarrow \frac{dp}{dt} > 0 \Rightarrow p \uparrow$$

For Kisa Vorobyaninov: If $p > \frac{1}{4} \Rightarrow \frac{dq}{dt} < 0 \Rightarrow q \downarrow$

$$\text{If } p < \frac{1}{4} \Rightarrow \frac{dq}{dt} > 0 \Rightarrow q \uparrow$$

Let us draw a phase plane (picture 2) to demonstrate what happens if players diverge from the equilibrium strategies.



* - stable equilibria

Picture 2. Phase diagram: equilibrium stability analysis

Arrows reflect changes in the probability of a certain choice in case the probability of the opponent's choice of a passive strategy diverges from the equilibria.

Let us make conclusions for the asymmetric game where there are different types of players. The players decide who can have a right of ownership by themselves. Only one of the players could have a right of ownership: either Ostap or Kisa. In the particular described case equilibrium is impossible if the object is in collective ownership of both players (as in case of a symmetric game).